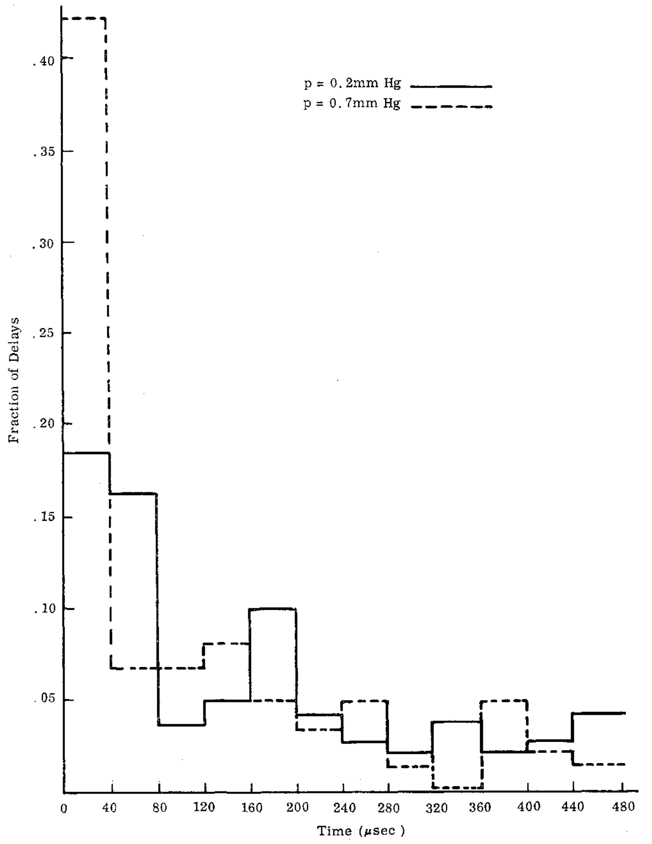

Fig. 1 Fraction of delays occurring in 40-μsec intervals.

It was at first suspected that the delays might be caused by discharge processes that limited current, after initiation of the discharge, to a small value, until some cathode electron emission process capable of supporting high current had time to establish itself. Therefore, a 1-meg resistance was placed in series between the electrodes and the capacitors. Even a small discharge current would produce an observable decrease in electrode voltage because of the voltage drop across the resistor. Thus, if the supposition had been correct, the resistor would have considerably shortened the observed delay in the voltage drop across the electrodes.

However, the results of these tests showed that no significant reduction in delay was produced by the resistor. Thus, the observed delays apparently were caused by statistical lags in the initiation of the discharges rather than lags in current buildup after initiation.

To further test this conclusion an auxiliary voltage supply was connected across the electrodes, so that a d.c. discharge of a few milliamperes was maintained between them. Under this condition, statistical lags were eliminated since a discharge was already in existence when the thyratron switch was fired. As expected, the delays under this condition were all very short, less than 10 μsec, and there was only small scatter in the recorded values.

The choice of aluminum, a metal commonly used for accelerator electrodes, may have reduced statistical lags considerably compared to those that might be found with other metals. Aluminum forms a thin, highly insulating oxide layer on its surface, and positive ions can collect on this layer. The resulting positive charge can produce sufficiently intense electric fields at the electrode surface to cause some cold cathode electron emission, and thus provide free electrons. This phenomenon, the "Malter" or "Paetow" effect (Ref. 3, p. 113), is familiar in Geiger counter work since electrons thus emitted sometimes cause spurious counts for several minutes after a count has been recorded. Therefore, since the interval between successive discharges was 30 sec, electrons emitted by this mechanism from oxide patches on the aluminum may have reduced statistical lags for the discharges following the initial one in a series. It would be useful to investigate whether electrodes constructed of metals that do not form insulating oxide layers, or the use of hydrogen propellant, which reduces oxides, can produce excessively long time delays in accelerators.

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Asymptotic Solution of a Toroidal Shell Subjected to Nonsymmetric Loads

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The determination of stresses and strains in a thin shell of revolution subjected to arbitrary loading reduces, according to Novozhilov's theory,[8] to integration of two differential equations in two complex stress function $\bar{U}_k(\Theta)\cos k\phi$ and $\bar{T}_k(\Theta)\cos k\phi$.

These equations, when specialized to a circular toroidal shell, acquire the following form:

$$G_k\left(\frac{\bar{U}_k}{r}\right) + \alpha k^2\left(\sin^2\Theta + \frac{1}{i\lambda^2}\frac{\alpha}{\rho}\right)\bar{T}_k = \frac{R}{\rho}g_k(\Theta) \quad \text{(1a)}$$

$$\frac{1}{i\lambda^2}G_k(\bar{T}_k) + \bar{T}_k\sin^2\Theta + \frac{1}{\rho r}\bar{U}_k = R\rho q_{n,k}(\Theta)\sin\Theta \quad \text{(1b)}$$

where

$$0 < \alpha = r/R < 1 \quad (2)$$

$$\lambda^2 = \frac{r^2}{Rt}[12(1-\nu^2)]^{1/2} \gg 1 \quad (3)$$

$$\rho = 1 + \alpha\sin\Theta \quad (4)$$

$$G_k(\ldots) = \frac{\sin^2\Theta}{\rho}\frac{d}{d\Theta}\left[\frac{\rho^2}{\sin\Theta}\frac{d(\ldots)}{d\Theta}\right] - \frac{k^2\alpha^2}{\rho}(\ldots)\sin\Theta \quad (5)$$

$$g_k(\Theta) = \sin^2\Theta(d/d\Theta)[\rho^3(q_{n,k}\cot\Theta - q_{1,k})] + k\alpha q_{2,k}\rho^2\sin^2\Theta \quad (6)$$

$q_{1,k}(\Theta)\cos k\phi \qquad q_{2,k}(\Theta)\sin k\phi \qquad q_{n,k}(\Theta)\cos_k\phi$

are the load components in the meridian, circumferential, and normal directions, respectively, ν is Poisson's ratio, and the remaining notation is explained in Fig. 1. The harmonic index k may have an integer value $n = 0, 1, 2, \ldots$ for shells with edges only at $\Theta = \text{const}$, as well as a noninteger one, $k = n(2\pi/\phi_0)$, for shells with edges at $\phi = 0$ and $\phi = \phi_0$ also.

Solutions of Eqs. (1) are known for the axisymmetrical case[3] $k = 0$ and for sinusoidal loading[4] $k = 1$ (see Ref. 2 for bibliography).

Moreover, parts of the toroid sufficiently distant from the top parallel circles, where the inequality

$$\lambda^2\sin^2\Theta \gg 1 \quad (7)$$

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holds, behave like any "regular" shell. Equations (1) may be separated here for arbitrary (not very large) values of k and the membrane and "edge" effects analyzed independently by the usual asymptotic methods developed for instance in Refs. 8 and 10.

Those parts of the shell where inequality (7) is not satisfied will be referred to as the shallow regions of the toroid.

In the present note an asymptotic method of solution for Eqs. (1) is proposed for the remaining cases, namely for the shallow regions of the toroid subjected to arbitrary, slowly-varying loads, excluding the symmetrical and sinusoidal loads.

We thus confine ourselves to the ranges

$$-(\pi/2) < \Theta < \pi/2 \tag{8a}$$

$$k^2/\lambda^2 \ll 1 \tag{8b}$$

$$k > 1 \tag{8c}$$

In these circumstances, Eqs. (1) cannot be separated. Eliminating the $\bar{U}_k$ function between them, introducing a new dependent variable

$$V = (1 + \alpha \sin\Theta)^2 \bar{T}_k \tag{9}$$

and disregarding second-order terms, we obtain the basic differential equation

$$L(V) = i\lambda^2 RF(\Theta) \tag{10}$$

where

$$L(V) = V'''' + i\lambda^2 \left\{ \frac{\sin\Theta}{\rho} V'' + \frac{3\cos\Theta}{\rho} V' + \left[\frac{\alpha(1-k^2)}{\rho^2} - \frac{2\sin\Theta}{\rho} \right] V \right\} \tag{11}$$

$$F(\Theta) = \frac{1}{\rho} [\rho^3 q_{n,k}]'' + \alpha q_{n,k}'\rho\cos\Theta - \alpha^2(k^2 - 3\cos^2\Theta) q_{n,k} + \frac{1}{\rho^2}(\rho^3 q_{1,k})' - \alpha k q_{2,k} \tag{12}$$

and the dashes denote differentiation with respect to Θ.

The particular integral V^* of Eq. (10) may be approximated from the membrane equation obtained from (10) by formally setting the thickness-to-radius ratio of the shell as zero. This corresponds to formally letting $\lambda \to \infty$ in (10) and leads to the equation

$$x(1-x^2)\frac{d^2V^*}{dx^2} + (3 - 4x^2)\frac{dV^*}{dx} - \left[\frac{\alpha(k^2-1)}{1+\alpha x} + 2x\right] V^* = (1 + \alpha x) RF(x) \tag{13}$$

where

$$x = \sin\Theta \tag{14}$$

The solution of Eq. (10) may be represented by the power series

$$V^* = a_0 + a_1 x + \ldots \tag{15}$$

convergent in the range (8a) considered, where the coefficients a_n are to be calculated by means of the usual methods, after the right-hand side of Eq. (13) has been expanded in power series in x.

The homogeneous equation

$$L(V_0) = 0 \tag{16}$$

corresponding to Eq. (10) contains a large parameter λ^2 and a "transition point" at $\sin\Theta = 0$. It belongs to the type of equations analyzed recently by Langer[5] and by Lin and Rabenstein.[7]

Employing the theory worked out by the latter authors[7] and proceeding in the manner demonstrated by Lin[6] on the example of Orr-Sommerfeld's equation of hydrodynamic stability, we obtain the general integral of Eq. (16) in the following form:

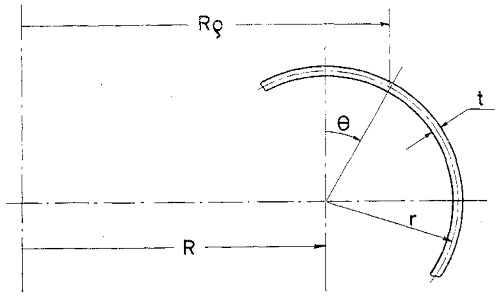

Fig. 1 Coordinate system and notation.

$$V_0 = A_0 u + A_1 \dot{u} + (1/i\lambda^2)(A_2 \ddot{u} + A_3 \dddot{u}) \tag{17}$$

In this formula, $u(z)$ is the known solution[9] of the equation

$$\dddot{u} + i\lambda^2(z\ddot{u} + 3\dot{u} + \beta u) = 0 \tag{18}$$

where

$$z = \pm \left[\frac{3}{2} \int_0^\Theta \left(\frac{|\sin\Theta|}{1 + \alpha\sin\Theta} \right)^{1/2} d\Theta \right]^{2/3} \tag{19}$$

$$\beta = \alpha k (k^2 - 1)^{1/2} \tag{20}$$

and dots denote differentiation with respect to z.

The plus sign preceding the brackets in formula (19) refers to positive values of $\sin\Theta$, the minus sign refers to negative Θ. The function $z(\Theta, \alpha)$ and its derivatives are tabulated in Ref. 2.

The coefficients A_i are given by the following formulas:

$$A_0 = -z\cdot|z|^{1/2}\beta^{-1/2}\{C_0\varphi_1(x)Y_3[2(\beta z)^{1/2}] + \varphi_2(x)J_3[2(\beta z)^{1/2}]\} \tag{21}$$

$$A_1 = -z^2\beta^{-1}\{C_0\varphi_1(x)Y_2[2(\beta z)^{1/2}] + \varphi_2(x)J_2[2(\beta z)^{1/2}]\} \tag{22}$$

$$A_2 = \frac{1}{z}\left[A_0 - 2A_3 + C_1\left(\frac{x}{z}\right)^{3/2}\left(\frac{dz}{d\Theta}\right)^{-5/2}\right] \tag{23}$$

$$A_3 = A_1/z \tag{24}$$

where the Bessel functions J_2, J_3, Y_2, and Y_3 of the first and second kind are to be replaced, respectively, by modified Bessel function $(-I_2)$, I_3, K_2, and K_3 for negative values of z.

Functions $\varphi_1(x)$ and $\varphi_2(x)$ represent the regular and singular solutions of the homogeneous membrane equation corresponding to Eq. (13):

$$x(1-x^2)\frac{d^2\varphi}{dx^2} + (3 - 4x^2)\frac{d\varphi}{dx} - \left[\frac{\alpha(k^2-1)}{1+\alpha x} + 2x\right]\varphi = 0 \tag{25}$$

They may be represented by power series with constant factors specified as follows:

$$\varphi_1(x) = 1 + (\alpha/3)(k^2 - 1)x + \ldots \tag{26}$$

$$\varphi_2(x) = \frac{1}{x^2}[1 - \alpha(k^2-1)x + \ldots] - \frac{\alpha^2}{2}k^2(k^2-1)\varphi_1(x)\ln|x| \tag{27}$$

The constants C_i,

$$C_0 = (\pi/2)\beta^2 \text{ (for } z \geqslant 0) \qquad C_0 = \beta^2 \text{ (for } z < 0)$$

$$C_1 = -1 \tag{28}$$

are so chosen that the coefficients A_i are all regular.

Now, in view of Eq. (9), the stress function $\bar{T}_k$ becomes

$$\bar{T}_k = (1 + \alpha \sin\Theta)^{-2}(V^* + V_0) \tag{29}$$

All the forces, moments and strain components can be determined in terms of the stress function $\bar{T}_k$ by means of the formulas given in Ref. 1.

The power series appearing in the solution converges slowly, as $|\sin\Theta|$ approaches unity. However inequality (7) then becomes valid and the aforementioned usual asymptotic methods can be applied. Both solutions must be matched along lines θ const common to two adjacent parts of the shell.

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Flow of Combustion Gases through a Perforation in a Solid Propellant Grain

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Nomenclature

A = duct or perforation cross-sectional area; A_w is the duct wall surface area; A^* is the critical cross-sectional area ($M = 1$)
F = frictional term $\triangleq [(\gamma/2)K(M_0)^2]$
K = geometry factor, Fig. 2
M = Mach number = velocity/$(g_c\gamma RT)^{1/2}$, where g_c is gravitational constant and T is static temperature
P = pressure; P_s is stagnation pressure
R = mixture gas constant = $R_0 m_g^*/\mu_g$; R_0 is universal gas constant; m_g^* is gas phase mass fraction; μ_g is gas phase molecular weight
w = mass flow rate; $w^* \triangleq w/w_i$
γ = isentropic path exponent that is the ratio of the mixture specific heats
τ = shear stress; $\tau_w dA_w$ is wall drag force, Fig. 1
σ $\triangleq \gamma M^2$
$\phi(M)$ = $M\{[2/(\gamma + 1)][1 + (½)(\gamma - 1)M^2]\}^{-(\gamma+1)/2(\gamma-1)}$ designated as A^*/A in standard gasdynamics table

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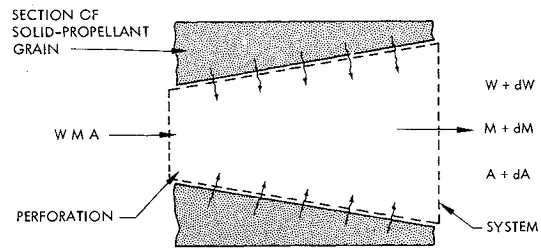

Fig. 1 Flow through a perforation in a solid propellant grain.

Subscripts

i, j = boundary condition
0 = value at smallest cross-sectional area at abrupt expansion or contraction, Fig. 2

Introduction

IN the design of solid propellant grains for solid rockets, the flow of combustion products inside perforations in the grains must be considered. This involves the analysis of a flowing gas stream with mass being added to it due to the burning propellant (Fig. 1). This flowing gas stream may also contain a condensed phase, as aluminum oxide, because of the burning of a metallized propellant.

A special case of this problem, involving the steady one-dimensional flow of a perfect gas through a constant cross-sectional area duct with mass addition along the duct and with a constant stagnation temperature, has been treated by several authors.[1,2] Wimpress[1] has solved for the properties along the duct using a Mach number dependent variable as the independent variable in his solution. Wimpress includes the frictional effects due to sudden expansions which may occur at the end of a solid propellant grain. Price[2] has solved for the properties along the duct using a nondimensional mass flow rate as the independent variable. As pointed out by Price, this choice of independent variable is more convenient to use than a Mach number dependent variable.

The analysis to follow considers the steady one-dimensional equilibrium flow of a perfect gas and condensed phase mixture in a duct whose cross-sectional area varies with length and whose mass flow rate increases with length. An approximate solution that lends itself to hand calculations is given. Eddy mixing effects due to abrupt expansions or contractions are also considered.

Analysis

The following idealizations apply to the system shown in Fig. 1: 1) the flow is one-dimensional and steady; 2) the main stream stagnation temperature is constant (the system is adiabatic and the propellant flame temperature is constant); 3) the mass addition contributes negligible momentum rate in the axial direction; 4) the wall drag force $\tau_w dA_w$ is in the axial direction; 5) the two phases are in equilibrium, and the mixture properties are constant; and 6) the condensed

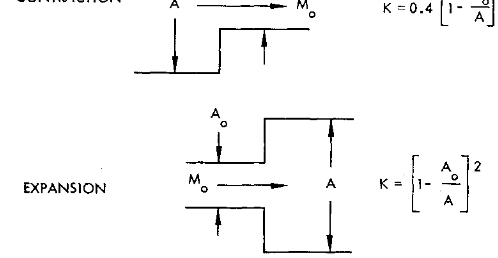

Fig. 2 Abrupt changes in perforation cross-sectional area.